k-colorable グラフの点彩色問題の近似アルゴリズムに対する議論

謝　旭珍、小野　孝男、平田　富夫
〒 464-8603 名古屋市千種区不老町
名古屋大学大学院工学研究科
電話: +81-52-789-3440
FAX: +81-52-789-3089
Email:
sharryx@hirata.nuee.nagoya-u.ac.jp, takao@hirata.nuee.nagoya-u.ac.jp, hirata@nuee.nagoya-u.ac.jp

概要
本論文では、グラフの点彩色問題とその近似アルゴリズムについて議論する。Wigderson は、$(\Delta+1)$ 色のアルゴリズムを部分アルゴリズムとして使って、$O(n^{1-1/(k-1)})$ 色のアルゴリズムを提案した。Karger らは、代わりに $\tilde{O}(\Delta^{1-2/k})$ 色アルゴリズムを部分アルゴリズムとして使って、$\tilde{O}(n^{1-3/(k+1)})$ 色のアルゴリズムを改良した。本論文では、$\tilde{O}(\Delta^{1-x/k})$ 色 $(2 < x < 3)$ アルゴリズムがあると仮定し、k-colorable グラフの点彩色問題に対してどのように改良できるか示す。

キーワード
点彩色問題、近似アルゴリズム、NP-困難、最大次数

Some Discussion on the Approximation Algorithms for Coloring k-colorable Graphs

XuZhen Xie, Takao Ono, Tomio Hirata
Graduate School of Engineering
Nagoya University
Furo-cho, Chikusa-ku
Nagoya 464–8603, Japan
Telephone: +81-52-789-3440
Fax: +81-52-789-3089
Email:
sharryx@hirata.nuee.nagoya-u.ac.jp, takao@hirata.nuee.nagoya-u.ac.jp, hirata@nuee.nagoya-u.ac.jp

Abstract

Wigderson presented an $O(n^{1-1/(k-1)})$-coloring algorithm for k-colorable graphs, which used an $(\Delta+1)$-coloring algorithm as a subroutine. Karger et al. improved the result to an $\tilde{O}(n^{1-3/(k+1)})$-coloring, using an $\tilde{O}(\Delta^{1-2/k})$-coloring algorithm as a subroutine instead. Suppose we have an algorithm using $\tilde{O}(\Delta^{1-x/k})$ colors, where $2 < x < 3$, how we can improve the results for k-colorable graphs.

Keywords

graph coloring, approximation algorithms, NP-hard, maximum degree

I. Introduction

A proper vertex coloring of a graph $G = (V, E)$ is an assignment of colors to its vertices such that no two adjacent vertices receive the same color. Equivalently, a k-coloring of G is a partition of its vertices into k independent sets. It is well known [1] [2] that the problem of properly coloring a k-colorable graph with k colors is NP-hard, for any $k \geq 3$.

Wigderson [3] gave a simple algorithm for coloring k-colorable graphs with $O(n^{1-1/(k-1)})$ colors. Blum [4] improved the result to $\tilde{O}(n^{a_k})$ colors, where $a_3 = 3/8, a_4 = 3/5, a_5 = 91/131, \cdots$. Karger, Motwani and Sudan [5] showed, using semidefinite programming, that k-colorable graph with maximum degree

Δ can be colored with $\tilde{O}(\Delta^{1-2/k})$ colors. Combined with the technique of Wigderson [3], Karger et al. [5] presented an $\tilde{O}(n^{1-3/(k+1)})$-coloring for k-colorable graph. Later, Blum and Karger [6] obtained an $\tilde{O}(n^{3/14})$-coloring for 3-colorable graph and most recently, Halperin, Nathaniel and Zwick [7] provided an $\tilde{O}(n^{b_k})$-coloring for $k \geq 4$, where $b_4 = 7/19, b_5 = 97/207, \cdots$.

The technique of Wigderson [3] leads to the interest with the maximum degree Δ. For example, Wigderson [3] and Karger et al. [5] only used different algorithms when the maximum degree is relatively small and obtained the different results in terms of n. The algorithm of Wigderson [3] uses $(\Delta+1)$-coloring algorithm for the graph with maximum degree Δ and colors the entire graph with $O(n^{1-1/(k-1)})$ colors. On the other hand, the algorithm of Karger et al. [5] uses $\tilde{O}(\Delta^{1-2/k})$-coloring algorithm for the graph with maximum degree Δ and leads to an $\tilde{O}(n^{1-3/(k+1)})$-coloring for the entire graph. Thus, we can easily deduce that if we have an algorithm using less colors in terms of Δ, then we can derive an algorithm using less colors in terms of n. In this paper, we assume that we have an algorithm that colors any k-colorable graph with maximum degree Δ using $\tilde{O}(\Delta^{1-x/k})$ colors where $2 < x < 3$, then how we could improve the results for k-colorable graph.

The rest of the paper is structured as follows. In Section 2, we give some definitions. In Section 3, we show the improvement over the algorithm of Karger et al. [5]. Section 4 an 5 are for the improvement over the algorithms of Blum and Karger [6] and Halperin et al.[7], respectively. Finally, concluding remarks is in Section 6.

II. Preliminaries and Definitions

Let us introduce the graph-theoretic notation that will be used throughout this paper. Given a graph G, let V denote the vertices of G and E denote the edges of G. We will use $N(v)$ to denote the neighborhood of a vertex v, $d(v)$ to denote the degree of v and Δ to denote the maximum degree of the graph. That is, for $G = (V, E)$,

$$\begin{aligned} N(v) &= \{u \mid (v,u) \in E\}, \\ d(v) &= |N(v)|, \\ \Delta &= \max_{v \in V} \{d(v)\}. \end{aligned}$$

The subgraph of G induced by $U \subseteq V$ is the graph $G_U = (U, F)$, where

$$F = \{(u,w) \mid u \in U, w \in U, \text{ and } (u,w) \in E\}.$$

III. The $\tilde{O}(n^{1-(x+t)/(k+1)})$-coloring

A. The Karger-Motwani-Sudan Algorithm

Karger, Motwani and Sudan [5] introduced the notion of *vector colorings* of a graph, which is closely related to Lovász's *orthogonal representations* and ϑ-function [8] [9] :

Definition ([5]) Given a graph $G = (V, E)$ on n vertices and a real number $k \geq 1$, a vector k-coloring of G is an assignment of n-dimensional unit vectors v_i to each vertex $i \in V$, such that for any two adjacent vertices i and j the dot product of their vectors satisfies the inequality:

$$\langle v_i, v_j \rangle \leq -\frac{1}{k-1}. \tag{1}$$

Karger, Motwani and Sudan [5] obtained the following results.

Theorem 1 ([5]) *Any k-colorable graph on n vertices with maximum degree Δ can be colored, in probabilistic polynomial time, using $\tilde{O}(\Delta^{1-2/k})$ colors.*

Theorem 2 ([5]) *Any k-colorable graph on n vertices can be colored, in probabilistic polynomial time, using $\tilde{O}(n^{1-3/(k+1)})$ colors.*

B. The $\tilde{O}(n^{1-(x+t)/(k+1)})$-coloring

Corollary 3 *Suppose we have an algorithm A that colors any k-colorable graph with maximum degree Δ using $\tilde{O}(\Delta^{1-x/k})$ colors, where $2 < x < 3$. Then we can derive an algorithm B that colors any k-colorable graph using $\tilde{O}(n^{A_k})$ colors, where $A_k \leq 1-(x+t)/(k+1)$, for $k \geq 3$ and $t = 0.927$.*

Proof. We use induction on k.

$k = 3$. While $\Delta \geq n^\rho$, let v be a vertex with $d(v) = \Delta$, 2-color the subgraph $G_{N(v)}$ induced by $N(v)$, set the colored vertices aside and repeat on the remaining graph using new colors. When $\Delta \leq n^\rho$, we apply algorithm A to color the remaining graph using $\tilde{O}(n^{(1-\frac{x}{3})\rho})$ colors. If

$$1 - \rho = (1 - \frac{x}{3})\rho, \tag{2}$$

then the 3-colorable graph can be colored with $\tilde{O}(n^{1-\frac{3}{6-x}})$ colors, where $2 < x < 3$.

Assume inductively that the claim is true for $(k-1)$-colorable graph. That is, algorithm B colors any $(k-1)$-colorable graph using $\tilde{O}(n^{A_{k-1}})$ colors. Now we will prove the inductive assertion of k.

While $\Delta \geq n^\rho$, apply algorithm B on the subgraph $G_{N(v)}$ induced by $N(v)$ of a vertex with $d(v) = \Delta$. Because $G_{N(v)}$ is $(k-1)$-colorable, algorithm B produces a coloring of $G_{N(v)}$ using $\tilde{O}(|N(v)|^{A_{k-1}})$ colors, from which an independent set of size $\tilde{\Omega}(|N(v)|^{1-A_{k-1}}) \geq \tilde{\Omega}(n^{(1-A_{k-1})\rho})$ is easily extracted. Giving the same color to all the vertices in this independent set, we set all the colored vertices aside and repeat on the remaining graph using new colors. When $\Delta \leq n^\rho$, we apply algorithm A to color the remaining graph using $\tilde{O}(n^{(1-\frac{x}{k})\rho})$ colors. Then algorithm B colors any k-colorable graphs using $\tilde{O}(n^{A_k})$ colors if the following equations hold:

$$A_k = (1 - \frac{x}{k})\rho, \tag{3}$$

$$(1 - \frac{x}{k})\rho = 1 - (1 - A_{k-1})\rho. \tag{4}$$

Soloving the equations with respect to A_k, we obtain the recurrence relation:

$$A_k = \frac{1 - \frac{x}{k}}{2 - \frac{x}{k} - A_{k-1}}. \tag{5}$$

We can rewrite this relation as follows:

$$\frac{1}{1 - A_k} = 1 + (1 - \frac{x}{k})\frac{1}{1 - A_{k-1}}, \tag{6}$$

and for $k = 3$,

$$A_3 = 1 - \frac{3}{6 - x}. \tag{7}$$

Let

$$\frac{1}{1 - A_k} = \frac{k + 1}{x + t}, \tag{8}$$

which is hold for $k = 3$, where $2 < x < 3$ and $t = 0.927$:

$$\frac{1}{1 - A_3} = \frac{6 - x}{3} < \frac{4}{x + t}. \tag{9}$$

Using equation (8) on the relation (6), we can prove that, where $2 < x < 3$ and $t = 0.927$,

$$A_k = 1 - \frac{x + t}{k + t} < 1 - \frac{x + t}{k + 1}. \tag{10}$$

The observation is that A_k is decreasing with x from (6), and the algorithm presented by Karger et al. [5] is the case as $x = 2$ (In this case, We can obtain that $A_k = 1 - 3/(k+1)$ from (6) with $A_3 = 1/4$.). Thus, if there were an algorithm that colors any k-colorable graph with maximum degree Δ using $\tilde{O}(\Delta^{1-x/k})$ colors, where $2 < x < 3$, then we could derive an algorithm using less colors than Karger et al. [5] for any k-colorable graph.

IV. The 3-colorable Graph

A. The Blum-Karger Algorithm

Applying Blum's algorithm (Theorem 13 of [4]) and combining with Karger et al. [5] for 3-colorable graph, Blum and Karger [6] provided the following results.

Lemma 4 ([6]) *In any* 3*-colorable graph with average degree exceeding* $2n^\rho$*, we can make progress towards an* $\tilde{O}(n^\alpha)$*-coloring where* $\alpha = \frac{3}{5}(1-\rho)$*.*

Theorem 5 ([6]) *There is a polynomial time algorithm to color any* 3*-colorable graph with* $\tilde{O}(n^{3/14})$ *colors.*

B. Combination of Blum's Algorithm and Algorithm A

We are given a 3-colorable graph. If its average degree is at least $2n^\rho$, we can color the graph using $\tilde{O}(n^\alpha) = \tilde{O}(n^{\frac{3}{5}(1-\rho)})$ colors based on Lemma 4. Otherwise, the graph has at least $\frac{n}{2}$ vertices of degree less than $4n^\rho$. The subgraph induced by those vertices clearly has maximum degree $\Delta \le 4n^\rho$, and we color the subgraph by algorithm A using $\tilde{O}(n^{(1-\frac{x}{3})\rho})$ colors. This coloring must contain an independent set of size $\tilde{O}(n^{1-(1-\frac{x}{3})\rho})$. Then we can color the 3-colorable graph using $\tilde{O}(\max\{n^{(1-\frac{x}{3})\rho}, n^{\frac{3}{5}(1-\rho)}\})$ colors. Let $\rho = \frac{9}{24-5x}$, we obtain the following result.

Corollary 6 *There is an algorithm to color any* 3*-colorable graph with* $\tilde{O}(n^{B_3}) = \tilde{O}(n^{\frac{9-3x}{24-5x}})$ *colors.*

We can rewrite B_3 as follows:

$$B_3 = \frac{3}{5}(1 - \frac{9}{24-5x}), \tag{11}$$

thus B_3 is decreasing with x. When $2 < x < 3$, it improves the result of $\tilde{O}(n^{\frac{3}{14}})$-coloring for 3-colorable graph.

V. A Look at Algorithm Combined-Color of Halperin et al.

A. Algorithm Combined-Color

By combining the coloring algorithms of Karger et al. [5], the combinatorial coloring algorithms of Blum [4], and an extension of a technique of Alon and Kahale [10] for finding relatively large independent sets in graphs, Halperin, Nathaniel and Zwick [7] obtained the new results.

Lemma 7 ([7]) *Let* $G = (V, E)$ *be a graph on* n *vertices with an independent set of size at least* n/k*, where* $k \ge 2$*. Then, a subset* $S \subseteq V$ *of size* $|S| \ge n/logn$*, and a vector* $(k + O(1/logn))$*-coloring of* G_S*, the subgraph of* G *induced by* S*, can be found in polynomial time.*

Lemma 8 ([7]) *Let* $G = (V, E)$ *be a* k*-colorable graph on* n *vertices. Then, an independent set of* G *of size* $\tilde{\Omega}(n^{f(k)})$ *can be found in polynomial time, where* $f(k) = 3/(k+1)$ *for* $k \ge 3$*.*

Theorem 9 ([7]) *Let* $G = (V, E)$ *be a* k*-colorable graph on* n *vertices that contains an independent set of size at least* n/k*. Then, an independent set of* G *of size* $\tilde{\Omega}(n^{f(k)})$ *can be found in polynomial time, where* $f(k) = 3/(k+1)$ *for* $k \ge 3$*.*

Theorem 10 ([7]) *Algorithm Combined-Color runs in polynomial time and it colors any* k*-colorable graph on* n *vertices using* $\tilde{O}(n^{\alpha_k})$ *colors, where* $\alpha_2 = 0, \alpha_3 = 3/14$*, and* $\alpha_k = 1 - \frac{6}{k+4+3(1-2/k)/(1-\alpha_{k-2})}$*, for* $k \ge 4$*.*

These results came from the following simple observation given by Blum [4]:

Lemma 11 ([4]) *Let* $k \ge 3$ *be an integer and* $0 < \alpha < 1$*. If in any* k*-colorable graph* $G = (V, E)$ *on* n *vertices we can find, in polynomial time, at least one of the following:*

1. *Two vertices* $u, v \in V$ *that have the same color under any valid* k*-coloring of* G *(Same Color),*
2. *An independent set* $I \subseteq V$ *of size* $\tilde{O}(n^{1-\alpha})$*(Large Independent Set),*

then, we can color every k*-colorable graph, in polynomial time, using* $\tilde{O}(n^\alpha)$ *colors.*

Theorem 12 ([4]) *Let* $G = (V, E)$ *be a* k*-colorable graph on* n *vertices with minimum degree* d_{min} *in which no two vertices have more than* s *common neighbors. Then, it is possible to construct, in polynomial time, a collection* $\mathcal{T}$ *of* $\tilde{O}(n)$ *subsets of* V*, such that at least one* $T \in \mathcal{T}$ *satisfies the following two conditions: (i)* $|T| \ge \tilde{\Omega}(\frac{d_{min}^2}{s})$*. (ii)* T *has an independent subset of size at least* $(\frac{1}{k-1} - O(\frac{1}{\log n}))|T|$*.*

B. Algorithm Combined-Color(x)

Using algorithm A, we can derive the following result similar to Lemma 8 [7] for finding a relatively large independent set in graphs.

Corollary 13 *Let $G = (V, E)$ be a k-colorable graph on n vertices. Then, an independent set of G of size $\tilde{\Omega}(n^{f(k)})$ can be found in polynomial time, where $2 < x < 3$, $t = 0.927$ and $f(k) = 1 - A_k \geq (x+t)/(k+1)$, for $k \geq 3$.*

Proof. The proof is by induction on k.

$k = 3$. We use the result in Section 3. Apply the derived algorithm B to color the 3-colorable graph using $\tilde{O}(n^{1-A_3}) = \tilde{O}(n^{1-\frac{3}{6-x}})$ colors, from which an independent set of size $\tilde{\Omega}(n^{\frac{3}{6-x}} \geq \tilde{\Omega}(n^{\frac{x+0.927}{4}})$ is extracted.

Assume inductively that for $(k-1)$-colorable graph, we can find an independent set of size $\tilde{\Omega}(n^{f(k-1)}) = \tilde{\Omega}(n^{1-A_{k-1}})$. Now for k-colorable graph, we describe two ways of finding independent sets of G. Using the algorithm A to find an independent set of size $\tilde{\Omega}(\frac{n}{\Delta^{1-x/k}})$. Alternatively, apply derived algorithm B on the subgraph $G_{N(v)}$ induced by $N(v)$ of a vertex v with $d(v) = \Delta$. Because $G_{N(v)}$ is $(k-1)$-colorable, then we can find an independent set of size $\tilde{\Omega}(\Delta^{f(k-1)})$. Taking the larger of these two independent sets, we obtain an independent set of G of size

$$\begin{aligned} &\tilde{\Omega}(\max\{\frac{n}{\Delta^{1-x/k}}, \Delta^{f(k-1)}\}) \\ \geq\ &\tilde{\Omega}\left(n^{\frac{1}{1+\frac{1-x/k}{f(k-1)}}}\right) = \tilde{\Omega}(n^{f(k)}). \end{aligned}$$

Similar to Corollary 3, we can prove that $f(k) = 1 - A_k \geq \frac{x+t}{k+1}$ as required.

Based on the Lemma 7 [7] and Corollary 13, we can obtain the following result, which is similar to Theorem 9 [7]. Here we omit the proof, which is the same as the one presented by Halperin et al. [7].

Corollary 14 *Let $G = (V, E)$ be a k-colorable graph on n vertices that contains an independent set of size at least n/k. Then, an independent set of G of size $\tilde{\Omega}(n^{f(k)})$ can be found in polynomial time, where $2 < x < 3$, $t = 0.927$ and $f(k) = 1 - A_k \geq (x+t)/(k+1)$, for $k \geq 3$.*

Now we derive the following algorithm Combined-Color(x) for $k \geq 4$.

Algorithm Combined-Color(x) for $k \geq 4$

1. Repeatedly remove from the graph G vertices of degree less than n^ρ. Let U be the set of vertices so removed and D be the average degree of G_U, thus $D \leq n^\rho$.

2. If $|U| \geq \frac{n}{2}$, apply algorithm A to find an independent set of G_U of size $\tilde{\Omega}(n/D^{1-x/k}) \geq \tilde{\Omega}(n^{1-(1-x/k)\rho})$. If $1-(1-x/k)\rho = 1 - B_k$, then we make progress of type 2.

3. Otherwise($|U| < \frac{n}{2}$), let $W = V - U$. Note that $|W| \geq \frac{n}{2}$ and that the minimum degree d_{min} in G_W satisfies $d_{min} \geq n^\rho$.

4. For every $u, v \in W$, consider the set $S = N(u) \cap N(v)$. If $|S| \geq n^{1-\beta}$, then apply the coloring algorithm recursively on G_S and $(k-2)$. If G_S is $(k-2)$-colorable, then the algorithm produces a coloring of G_S using $\tilde{O}(|S|^{B_{k-2}})$ colors, from which an independent set of size $\tilde{\Omega}(|S|^{1-B_{k-2}}) \geq \tilde{\Omega}(n^{(1-B_{k-2})(1-\beta)})$ is easily extracted. If $(1-B_{k-2})(1-\beta) = 1 - B_k$, then we make progress of type 2. If the coloring returned by the recursive call uses more than $\tilde{O}(|S|^{B_{k-2}})$ colors, we can infer that G_S is not $(k-2)$-colorable and thus, u and v must be assigned the same color under any valid k-coloring of G, then we make progress of type 1.

5. Otherwise we know that $|S| < n^{1-\beta}$ for every $u, v \in W$. Also, we know that the minimum degree d_{min} in G_W satisfies $d_{min} \geq n^\rho$.

6. We now apply Blum's algorithm, with $d_{min} \geq n^\rho$ and $s < n^{1-\beta}$, and obtain a collection $\mathcal{T}$ of $\tilde{O}(n)$ subsets of W such that at least one $T \in \mathcal{T}$ satisfies $|T| \geq \tilde{\Omega}(\frac{d_{min}^2}{s}) \geq \tilde{\Omega}(n^{2\rho+\beta-1})$, and T contains an independent set of size at least $(\frac{1}{k-1} - O(\frac{1}{\log n}))|T|$.

7. Now apply the result of Corollary 14 on G_T, for each $T \in \mathcal{T}$. In at least one of these runs we obtain an independent set of size $\tilde{\Omega}(n^{(1-A_{k-1})(2\rho+\beta-1)})$. If $(1-A_{k-1})(2\rho+\beta-1) = 1-B_k$, then we make progress of type 2.

Algorithm Combined-Color(x) colors any k-colorable graph with $\tilde{O}(n^{B_k})$ colors if the following equations hold.

$$
\begin{aligned}
1-(1-\frac{x}{k})\rho &= 1-B_k, &(12)\\
(1-B_{k-2})(1-\beta) &= 1-B_k, &(13)\\
(1-A_{k-1})(2\rho+\beta-1) &= 1-B_k. &(14)
\end{aligned}
$$

Solving these equations with respect to B_k, we obtain the recurrence relation:

$$
B_k = \frac{1+\frac{1-A_{k-1}}{1-\alpha_{k-2}}}{1+(1-A_{k-1})(\frac{2}{1-x/k}+\frac{1}{1-\alpha_{k-2}})}. \tag{15}
$$

We can rewrite this relation as follows:

$$
\frac{1}{1-B_k} = \frac{1}{2}\left\{2+\frac{1-x/k}{1-A_{k-1}}+(1-\frac{x}{k})\frac{1}{1-B_{k-2}}\right\}. \tag{16}
$$

and $B_2 = 0$ for $k = 2$, $B_3 = (9-3x)/(24-5x)$ for $k = 3$. It is easy to observe that B_k and A_{k-1} are decreasing with x. When $2 < x < 3$, we can improve the result of Halperin et al. [7].
Using the result of Corollary 3, let

$$
A_k = 1-\frac{x+t}{k+1}, \tag{17}
$$

and rewrite (16) as follows:

$$
\frac{1}{1-B_k} = \frac{1}{2}\left\{1+\frac{k+t}{x+t}+(1-\frac{x}{k})\frac{1}{1-B_{k-2}}\right\}. \tag{18}
$$

It can be proved that for $2 < x < 3$, $t = 0.927$ and $k \geq 4$, the following equation holds:

$$
\frac{1}{1-B_k} = \frac{k+t-1}{x+t}+\frac{x(3-t)}{k(x+t)}+C_k, \tag{19}
$$

where C_k satisfies the following recurrence relation:

$$
\begin{aligned}
C_k = \frac{x(2-x)(3-t)}{2k(k-2)(x+t)} &+ \frac{k-x}{2k}C_{k-2}\\
&\leq \frac{1}{2}C_{k-2}.
\end{aligned}
$$

We can obtain that

$$
B_k = 1-\frac{x+t}{k+t-1}+O(\frac{1}{k^2}). \tag{20}
$$

VI. Concluding remarks

If there were an algorithm that colors any k-colorable graph with maximum degree Δ using $\tilde{O}(\Delta^{1-x/k})$ colors where $2 < x < 3$ and $k \geq 3$, we have derived some improved results for k-colorable graph. The remaining interesting problem is how to find such an algorithm.

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